Math 1653W

Wednesday 25 October 2023

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One possible answer is
$y = \frac{1}{2}x + \frac{7}{2}$,
but this takes a little work (to find $b = 7/2$).

Warm-up: Give an equation for the line through the point $(3, 5)$ with slope $\frac{1}{2}$.

Easier answer:
$y = 5 + \frac{1}{2}(x-3)$

See tasks 8–15 from List 0.

Limits of functions

Last time

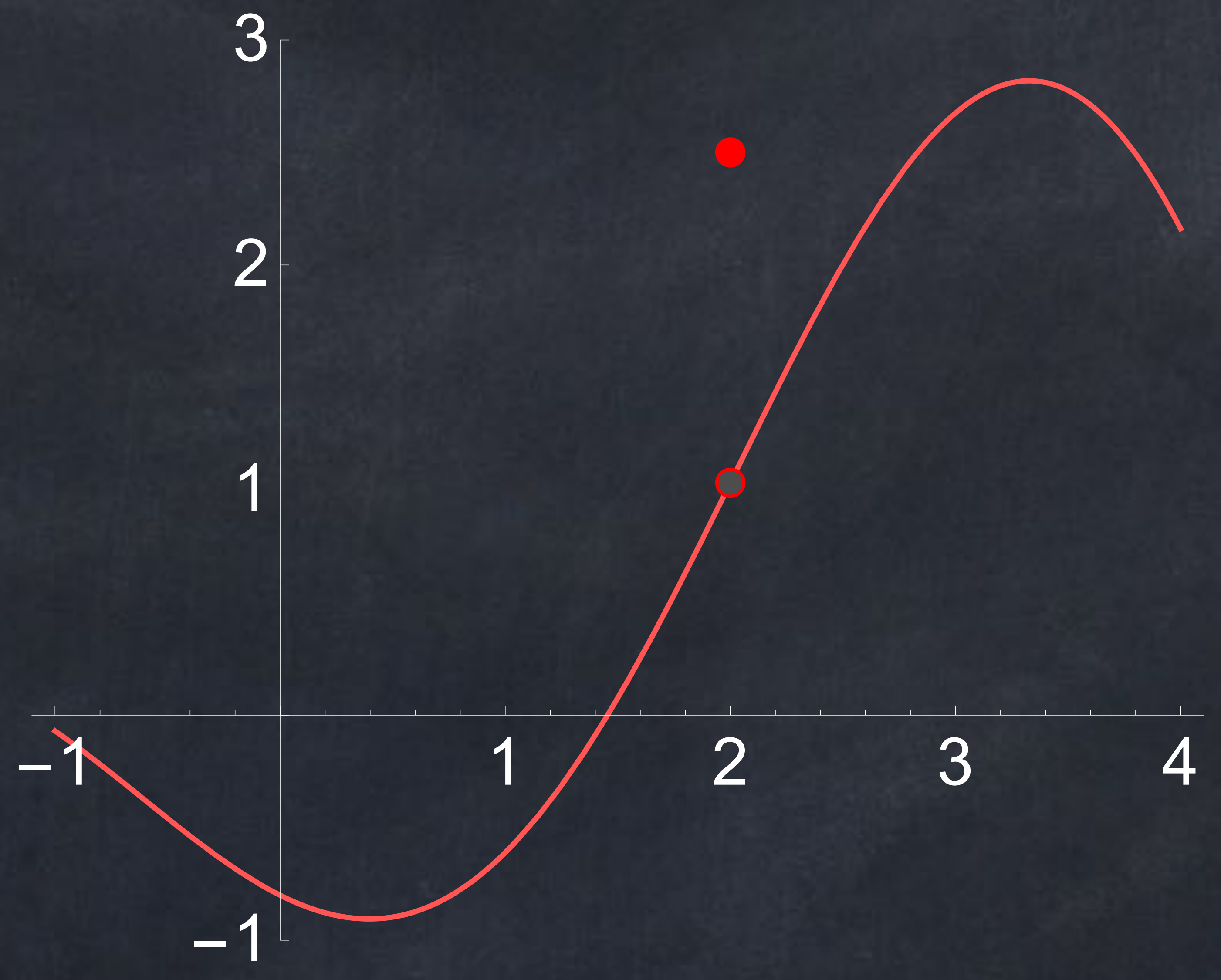
For functions, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are very similar to limits of sequences.

Functions can also have limits *at a point*.

- $\lim_{x\to a} f(x)$ is a two-sided limit, also just called a limit.
- $\lim_{x\to a^-} f(x)$ is a limit from the left ($x < a$), also called from below.
- $\lim_{x\to a^+} f(x)$ is a limit from the right ($x > a$), also called from above.

If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ have different values, or if at least one of them does not exist, then $\lim_{x\to a} f(x)$ does not exist.

It's possible for $\lim_{x \to a} f(x)$ to be completely unrelated to the value $f(a)$.



But for many functions we *can* calculate limits just by finding the value of the function at the point:

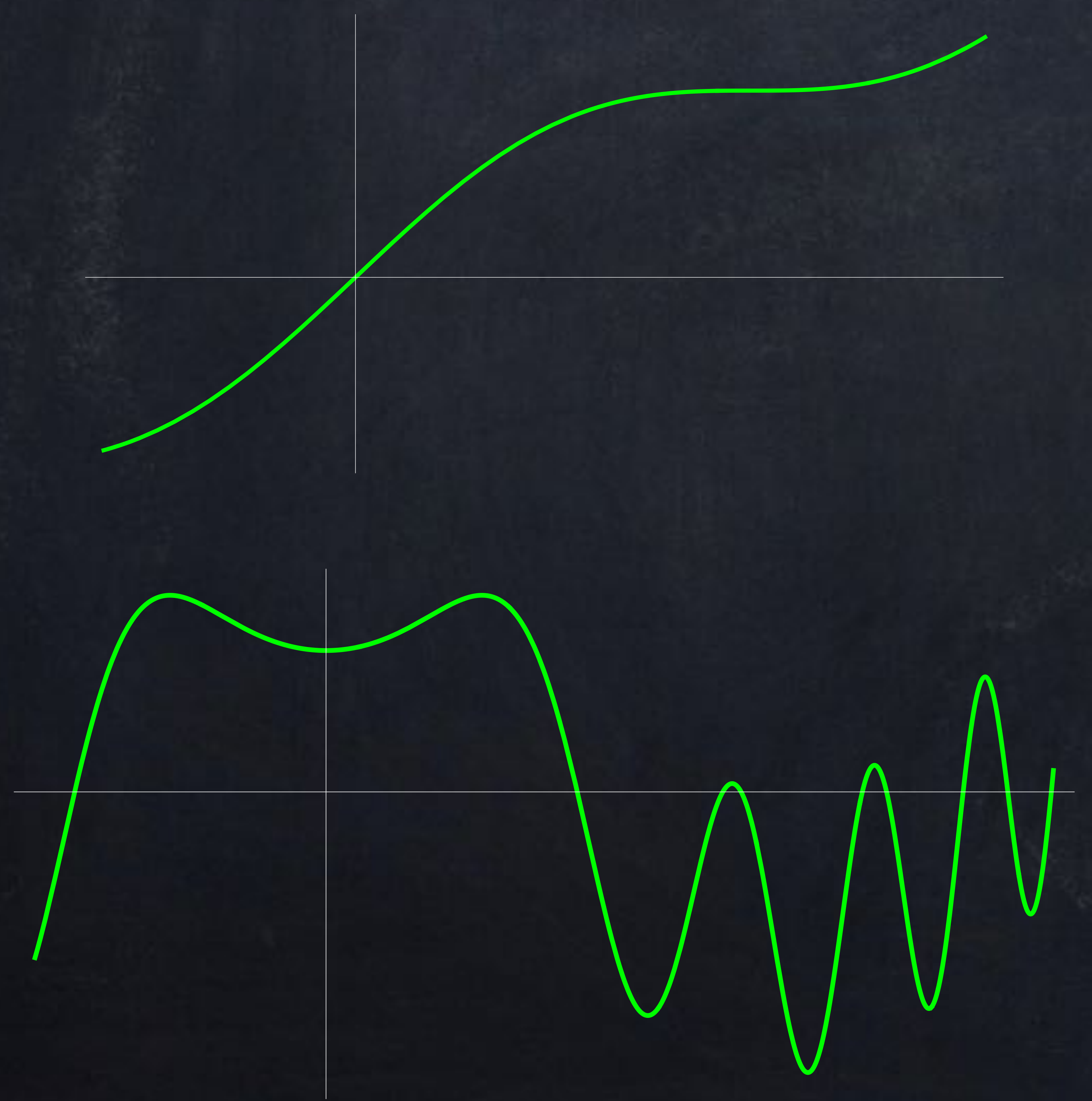
$$\lim_{x \to 7} \frac{x^2+1}{x^2-2} = \frac{7^2+1}{7^2-1} = \frac{50}{47}.$$

When exactly this is allowed brings us to the idea of “continuity”.

Continuity

Informally, a **continuous function** is one with no holes, jumps, or asymptotes in its graph. This means you could draw its graph without picking up your pen or pencil.

Continuous



Not continuous

Continuity

Let $f(x)$ be a function and let p be a number.

We say "**f is continuous at p**" if all of these are true:

1. $f(p)$ is defined,
2. $\lim_{x \to p} f(x)$ exists,
3. $\lim_{x \to p} f(x) = f(p)$.

If any of these is false, f is **discontinuous**.

What can discontinuity look like?

Types of discontinuities

The graph $y = f(x)$ has a **jump** at $x = a$ if

1. $\lim_{x \to a^-} f(x)$ is finite, and
2. $\lim_{x \to a^+} f(x)$ is finite, and
3. $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

(drawings on board during class)

The graph $y = f(x)$ has a **hole** at $x = a$ if

1. $\lim_{x \to a} f(x)$ is finite and
2. $f(a)$ is not defined or $f(a) \neq \lim_{x \to a} f(x)$.

Types of discontinuities

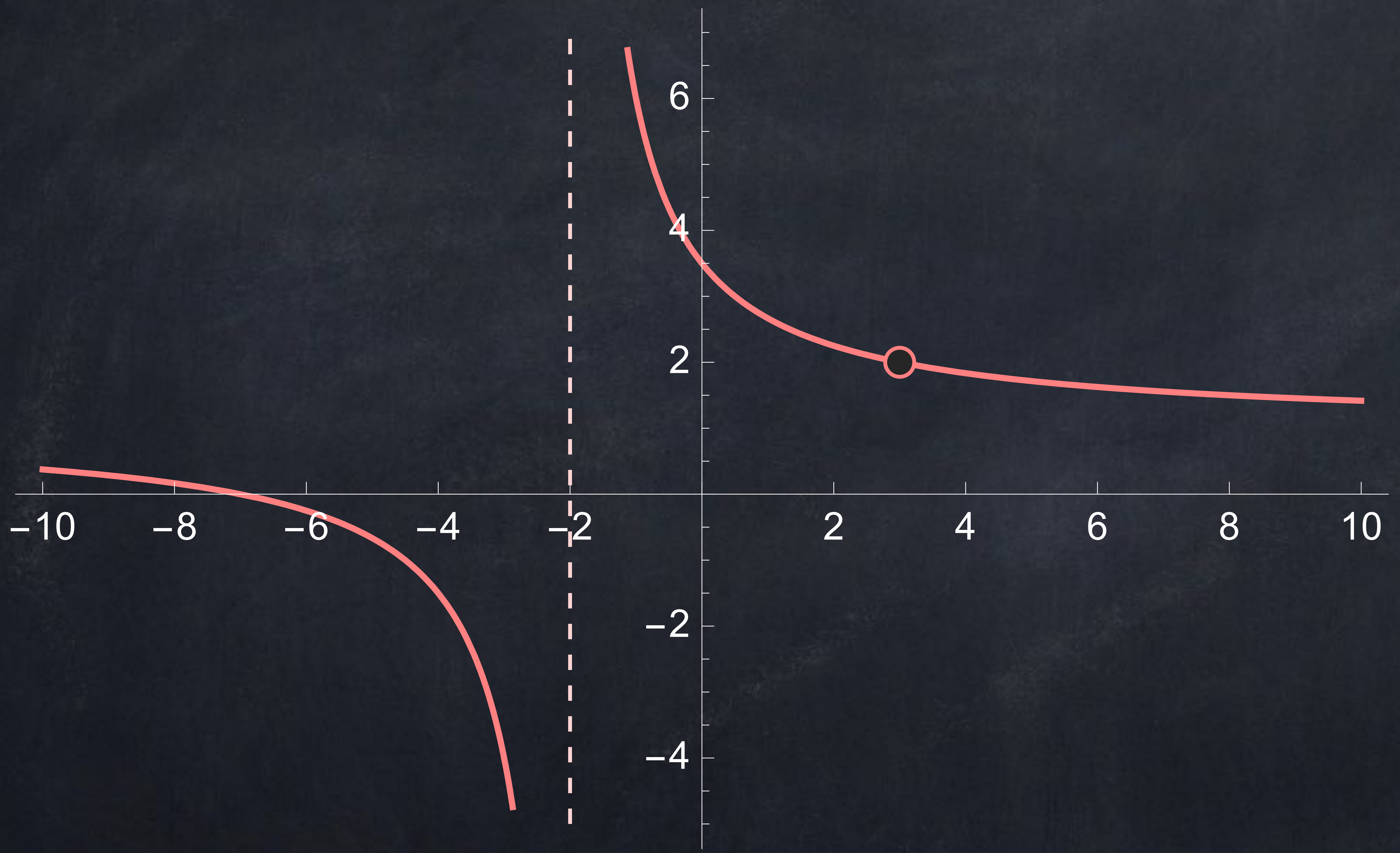
The vertical line $x = a$ is a **vertical asymptote** of the graph $y = f(x)$ if *at least one* of the following are true:

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty \quad \text{or}$$

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = +\infty.$$

Note: a horizontal asymptote ($\lim_{x \to \infty} = L$ or $\lim_{x \to -\infty} = L$) has nothing to do with discontinuity.

Task 1: *Describe* the discontinuities of $f(x) = \dfrac{(x+7)(x-3)}{(x+2)(x-3)}$.



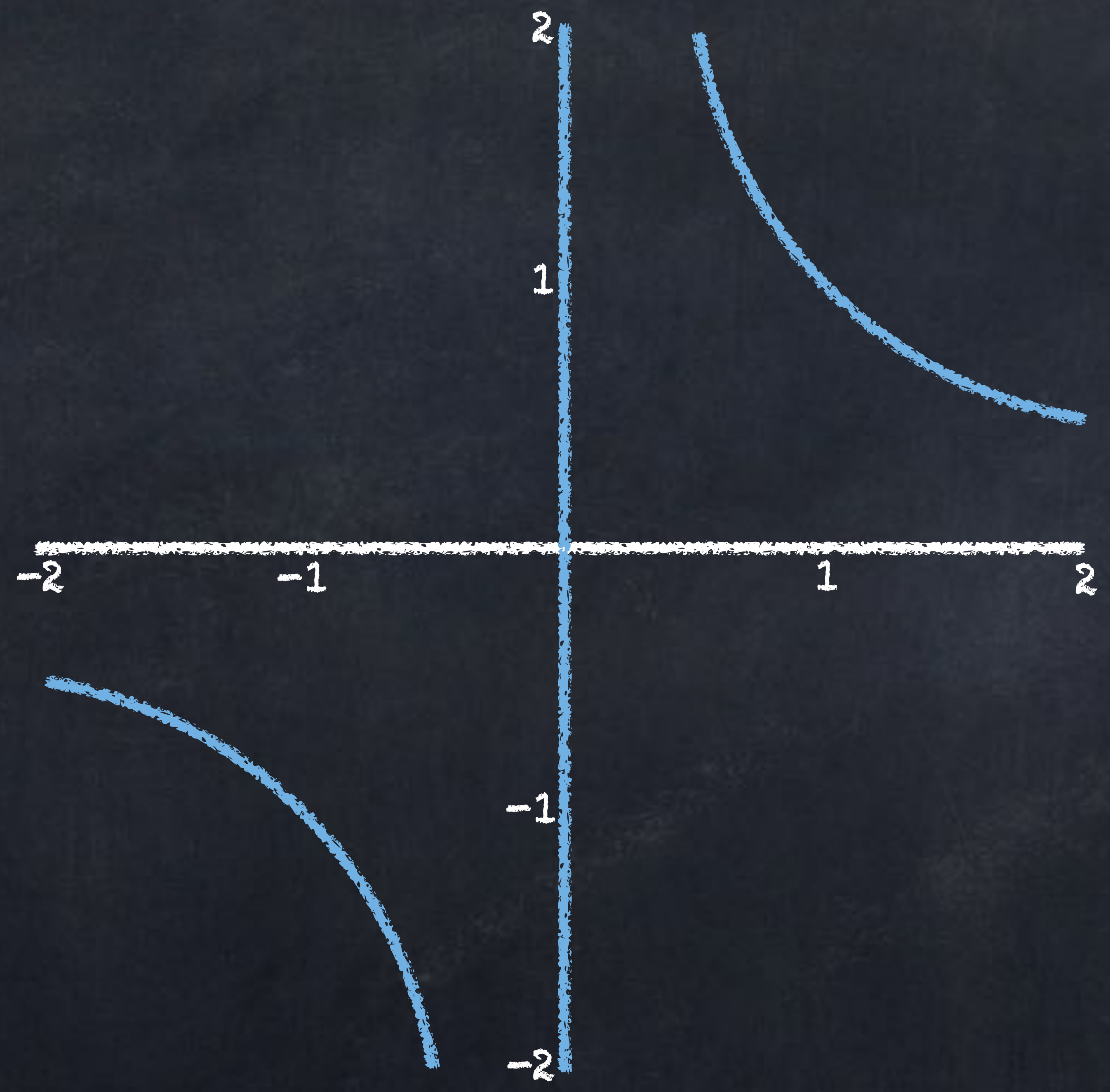
Hole at x = 3.

Vertical asymptote at x = −2.

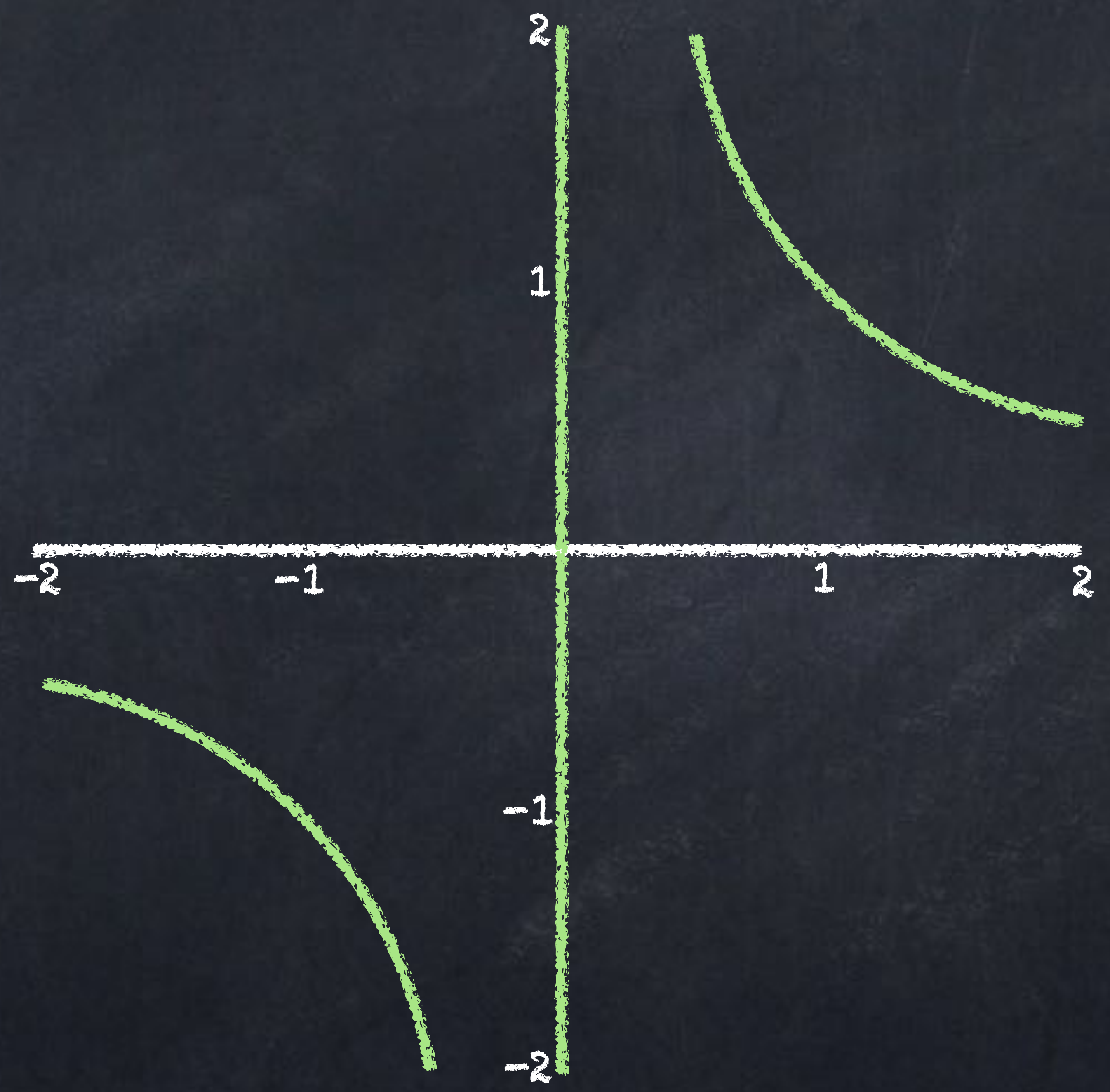
(There is also a horizontal asymptote at y = 1 and a "zero" at x = −7, but the task does not ask about these.)

Why talk about limits when we can just ask a computer to graph a function?

$$y = \frac{100x^2 + 100}{100x^3 + 100x}$$

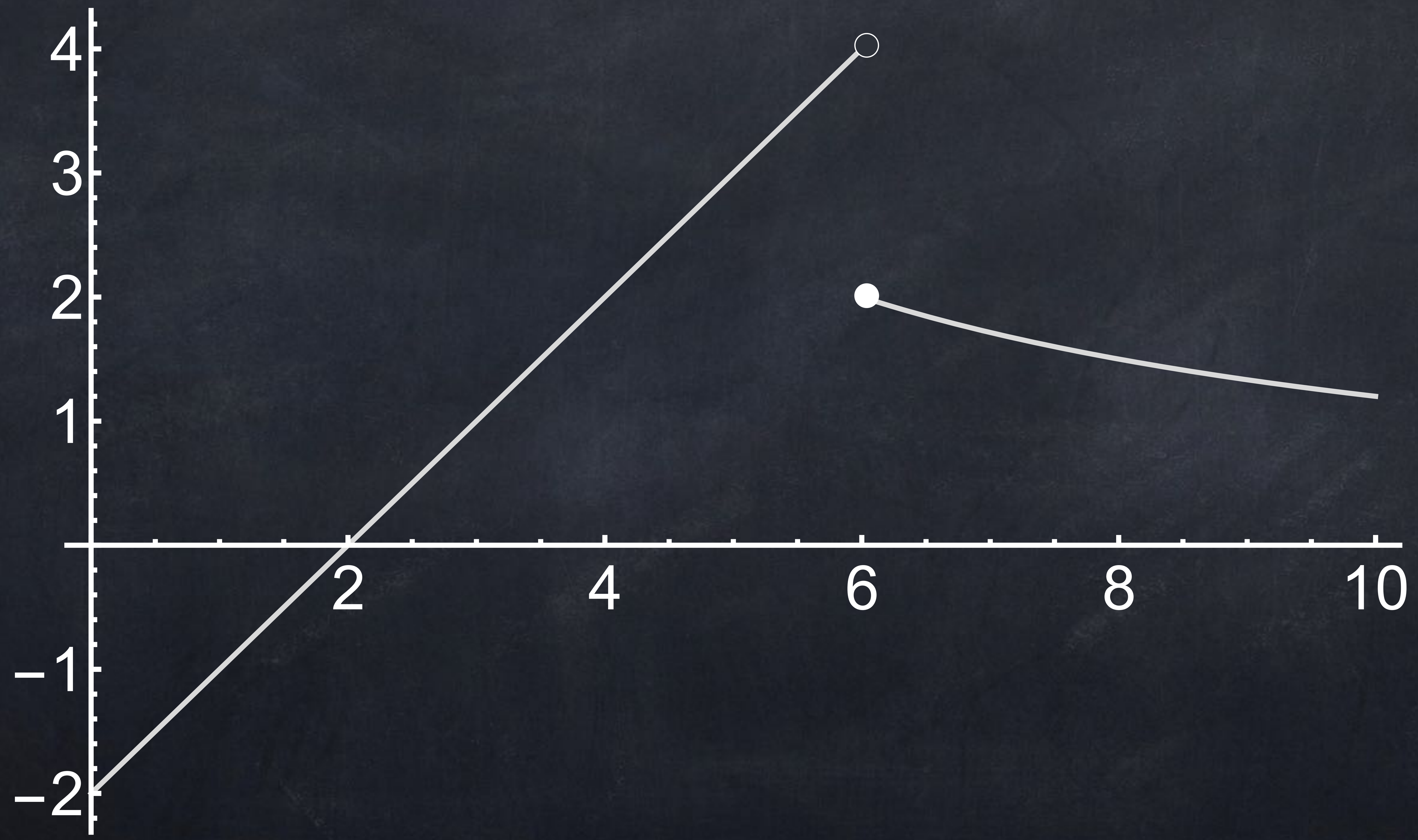


$$y = \frac{100x^2}{100x^3 + x}$$



Using limits, we can see that the left graph has an asymptote, right has a hole.

Task 2: What is $\lim_{x \to 6^-} f(x)$ for the function below?



Continuity

Our previous definition describes when a function is *continuous at a point*.

We say "**f is continuous on the interval $[a, b]$**" if it is continuous at all points p for which $a \leq p \leq b$.

- We can also talk about open intervals (a, b), semi-open intervals $[a, b)$ or $(a, b]$, and infinite intervals like $[a, \infty)$.

We say "**f is continuous everywhere**" or "**f is continuous**" or "**f is a continuous function**" if it is continuous at all points.

Continuity

Important examples:

- Any polynomial is continuous.
  - This includes constant functions.
- $\sqrt[n]{x}$ is continuous if n is odd.
- $\sqrt[n]{x}$ is continuous on $[0,\infty)$ if n is even.
- $\sin(x)$ and $\cos(x)$ are continuous.
- e^x and a^x are continuous.
- $\ln(x)$ and $\log_b(x)$ are continuous on $(0,\infty)$.

You can use all of these without giving any proofs or reasons.

We saw this exact list last week when talking about limit properties.

Limit properties

Last time

If the limits all exist and are finite, then

- $\lim_{x\to a}\Big(f(x)+g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big) + \Big(\lim_{x\to a} g(x)\Big)$,
- $\lim_{x\to a}\Big(f(x)\cdot g(x)\Big) = \Big(\lim_{x\to a} f(x)\Big)\Big(\lim_{x\to a} g(x)\Big)$,
- $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$,
- $\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big)$ if f is a ~~"nice"~~ **continuous** function.

These all work for $x\to\infty$, $x\to-\infty$, $x\to a^+$, and $x\to a^-$ too!

Continuity

Using the limit rules individually, we have

$$\lim_{x\to 4} \frac{e^{\sin(\pi x)}}{1+x^2} = \frac{\lim_{x\to 4} e^{\sin(\pi x)}}{\lim_{x\to 4}(1+x^2)} = \frac{e^{\left(\lim_{x\to 4}\sin(\pi x)\right)}}{1+\lim_{x\to 4} x^2}$$

$$= \frac{e^{\sin\left(\pi \lim_{x\to 4} x\right)}}{1+\left(\lim_{x\to 4} x\right)^2} = \frac{e^{\sin(4\pi)}}{1+4^2} = \frac{e^0}{1+16} = \frac{1}{17}$$

But we do not need to go into this much detail every time we write the answer to a limit.

Continuity

Fact: if $f(x)$ and $g(x)$ are both continuous, then the

- **sum** $f(x) + g(x)$,
- **difference** $f(x) - g(x)$,
- **product** $f(x) \cdot g(x)$, and
- **composition** $f(g(x))$

are all continuous.

- If $g(x)$ is never 0 then the **quotient** $\frac{f(x)}{g(x)}$ is also continuous.

Example: $\lim_{x \to 4} \frac{e^{\sin(\pi x)}}{1 + x^2} = \frac{e^{\sin(4\pi)}}{1 + 4^2} = \frac{1}{17}$.

Intermediate Value Theorem (IVT)

If f is continuous on $[a, b]$ then for any number Y between $f(a)$ and $f(b)$ there exists a number X between a and b for which $f(X) = Y$.

Example application: Can we find a number $0 \leq x \leq 1$ for which $2^x = 6x$ using only a simple calculator (+−×÷)?

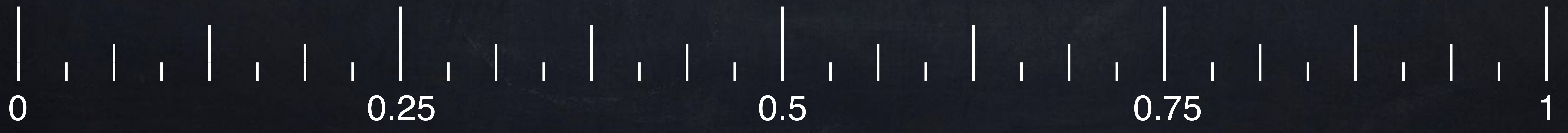
- First, does such a number exist?
  - Let $f(x) = 2^x - 6x$. This is continuous on $[0,1]$.
  - $f(0) = 1$ and $f(1) = -4$.
  - Because 0 is between -4 and 1, the IVT tells us there must be some $0 \leq x \leq 1$ with $f(x) = 0$.

Can we find a number $0 \leq x \leq 1$ for which $2^x = 6x$ using only a simple calculator?

- Let $f(x) = 2^x - 6x$. This is continuous!

$f(0.5) = -1.59 < 0$, so $f(x)=0$ for some x in $[0,0.5]$.
$f(0.25) = -0.31 < 0$, so $f(x)=0$ for some x in $[0,0.25]$.
$f(0.125) = 0.34 > 0$, so $f(x)=0$ for some x in $[0.125,0.25]$.

After more guesses in this way, we can find that x is approximately 0.1901.



Speed (or velocity)

- If you run $114\,\text{km}$ in 5 hours, what is your "average speed"?

- If your position in meters after t seconds is

$$P(t) = \frac{1}{10}t^2 + 2t,$$

calculate your "average speed" (in m/s) between $t = 2$ and $t = 10$.

$$\frac{P(10) - P(2)}{10 - 2} = \frac{30 - 4.4}{8} = 3.2\,\frac{m}{s}$$

Speed (or velocity)

- If your position after t seconds is

$$P(t) = \frac{1}{10}t^2 + 2t,$$

estimate your "*instantaneous* speed" when $t = 2$.

$$\frac{P(2.1) - P(2)}{0.1} = \frac{4.641 - 4.4}{0.1} = 2.41\,\frac{m}{s}$$

$$\frac{P(2.001) - P(2)}{0.001} = 2.4001\,\frac{m}{s}$$

Speed (or velocity)

- If your position after t seconds is

$$P(t) = \frac{1}{10}t^2 + 2t,$$

calculate your "*instantaneous* speed" exactly when $t = 2$.

$$\lim_{t \to 2} \frac{P(t) - P(2)}{t - 2} \quad \text{or} \quad \lim_{h \to 0} \frac{P(2+h) - P(2)}{h}$$

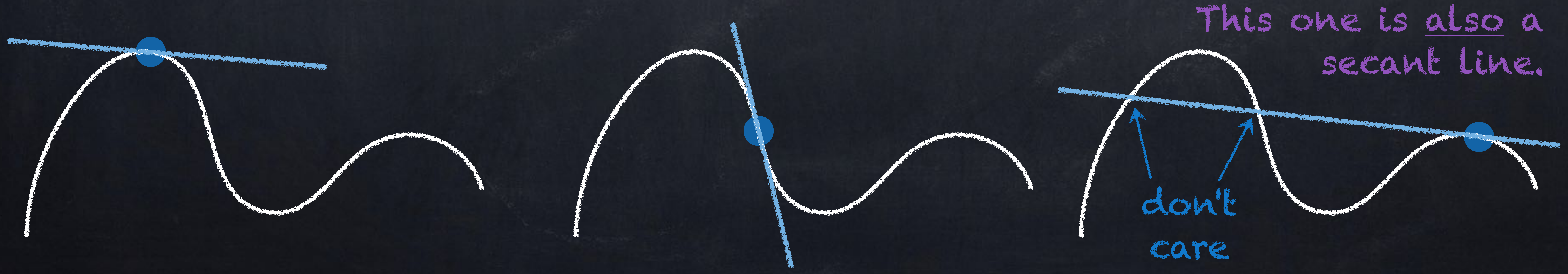
New variable $h = t - 2$ is the amount of time after 2 sec.

$t = 2 + h$

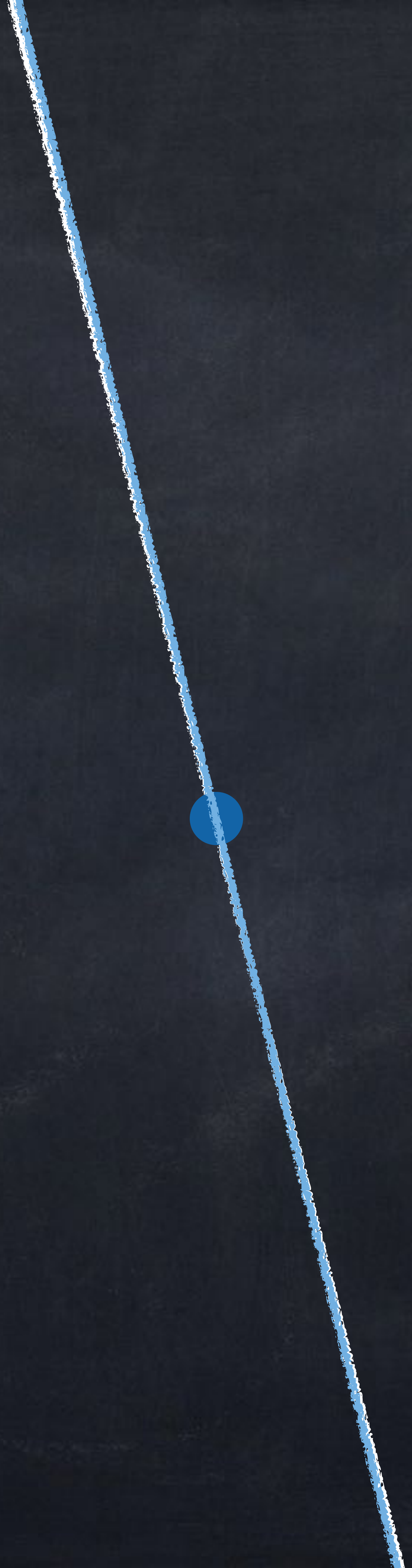
Secant lines and tangent lines

A **secant line** is a line that intersects a curve at two or more points.

A **tangent line to a curve at a point** is a line that "touches" the curve at that point (it might or might not intersect the curve again farther away).



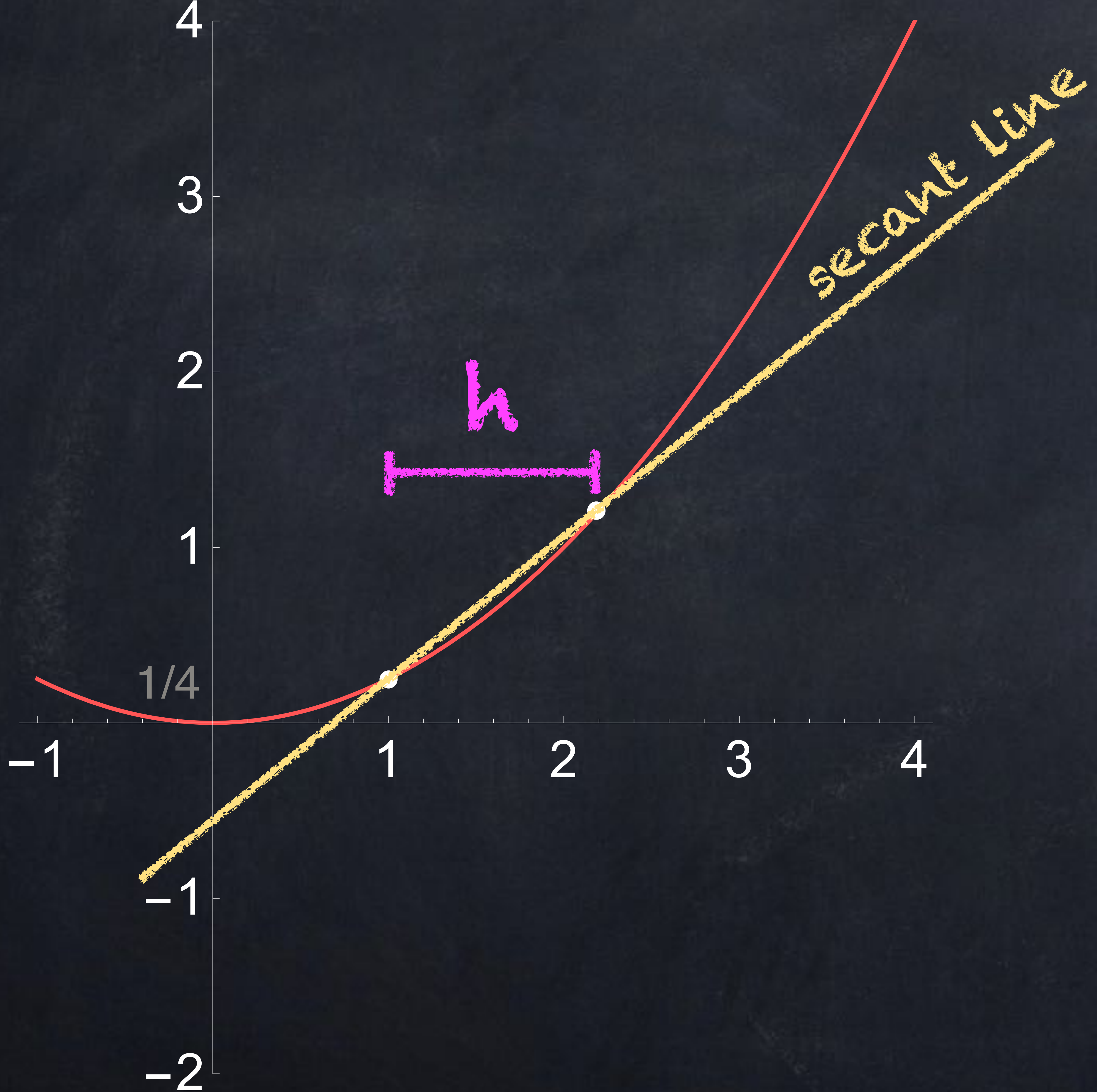
After “zooming in” to a
picture with a tangent line:

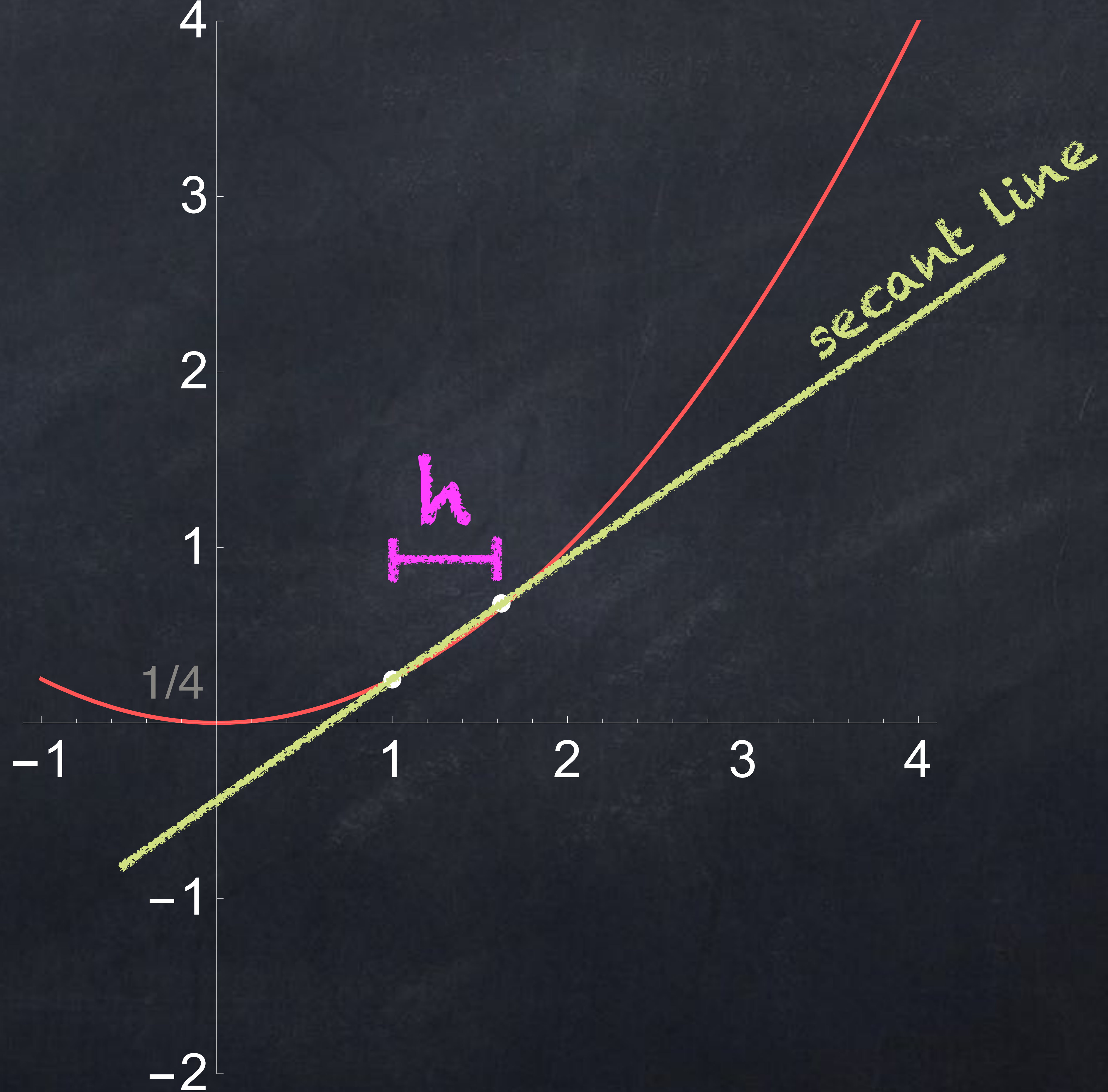


A tangent line at p is <u>very close</u> to the original curve near $x = p$.

The idea of a tangent line is *not* about the number of intersections.

What is the slope of the tangent line to $y = \frac{x^2}{4}$ at point $(1, \frac{1}{4})$?





What is the slope of the <u>tangent line</u> to $y = \dfrac{x^2}{4}$ at point $(1, \frac{1}{4})$?

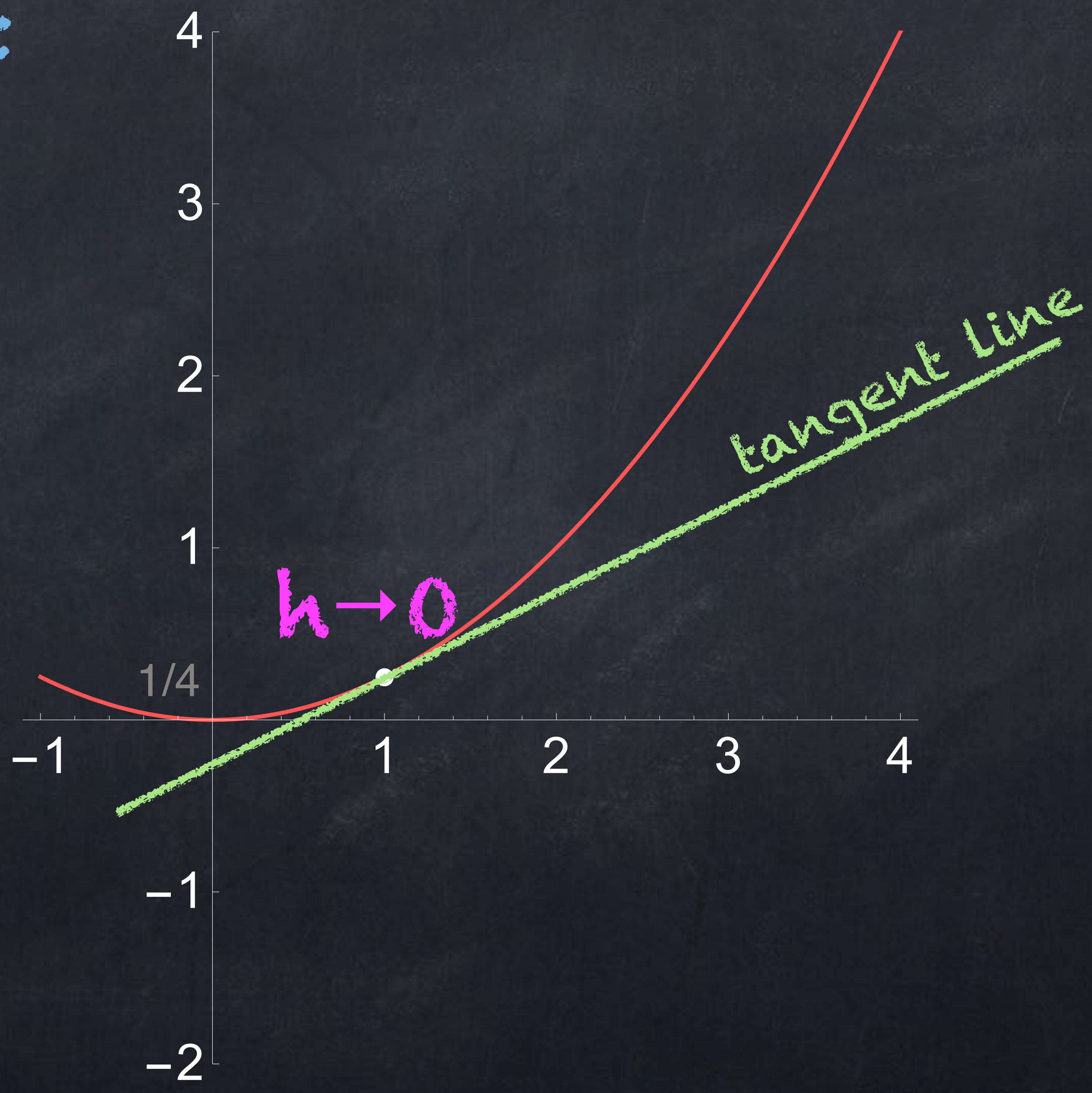
We can use either of these limits:

$$\text{slope} = \lim_{x \to 1} \frac{\frac{1}{4}(x)^2 - \frac{1}{4}}{x - 1}$$

$h = x - 1$

$x = h + 1$

$$\text{slope} \lim_{h \to 0} \frac{\frac{1}{4}(1+h)^2 - \frac{1}{4}}{h}$$

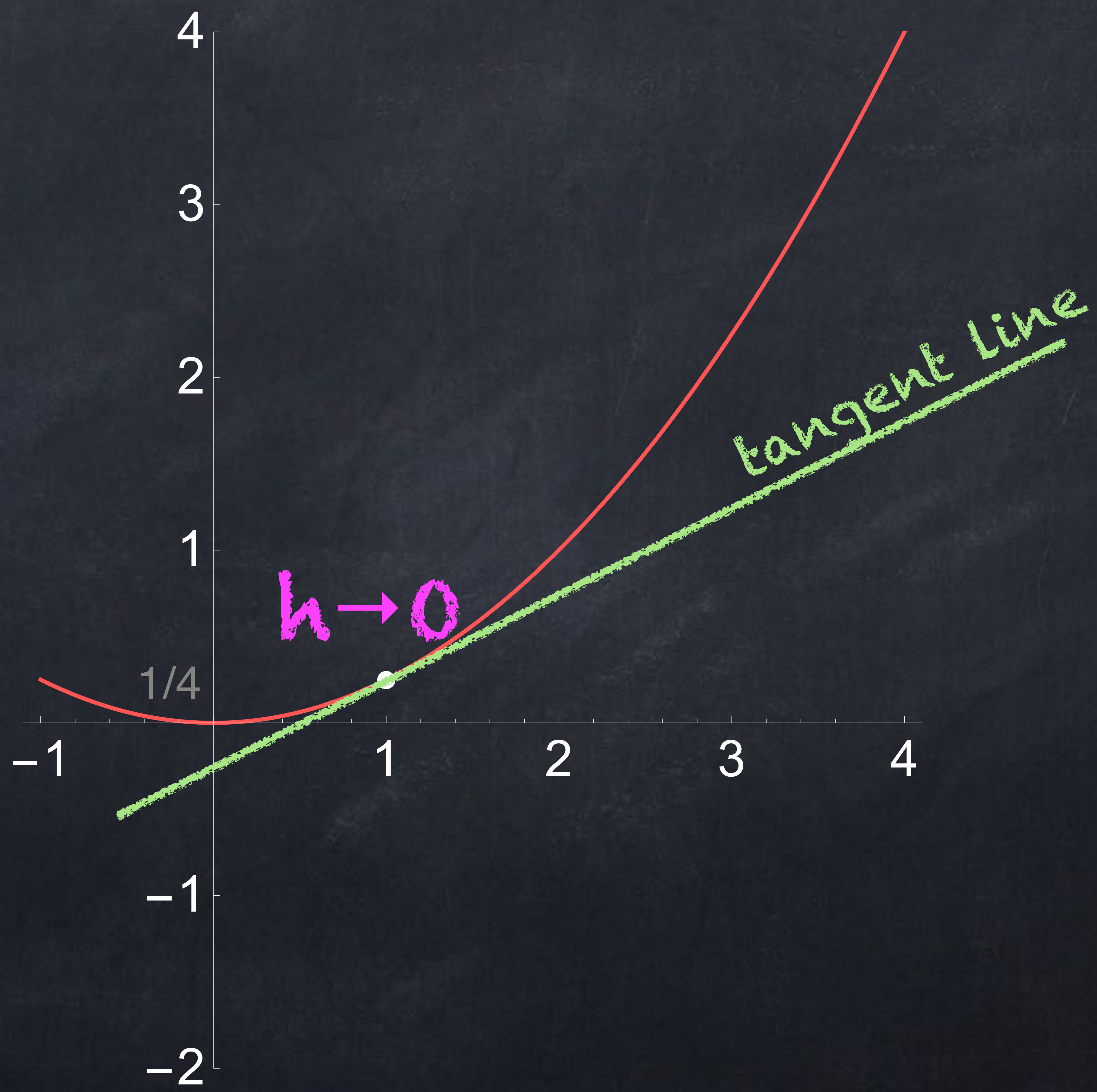


What is the slope of the tangent line to $y = \frac{x^2}{4}$ at point $(1, \frac{1}{4})$?

$$\lim_{h\to 0} \frac{\frac{1}{4}(1+h)^2 - \frac{1}{4}}{h}$$

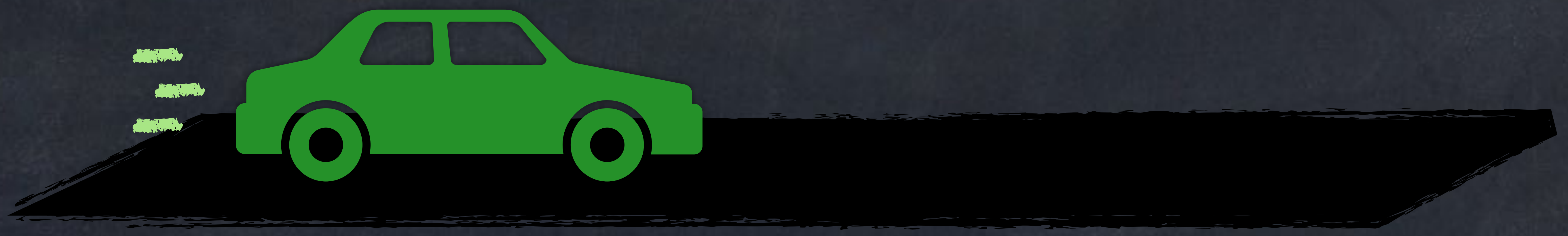
$$= \lim_{h\to 0} \frac{\left(\frac{1}{4} + \frac{2}{4}h + \frac{1}{4}h^2\right) - \frac{1}{4}}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{2} + \frac{1}{4}h\right) = \boxed{\frac{1}{2}}$$

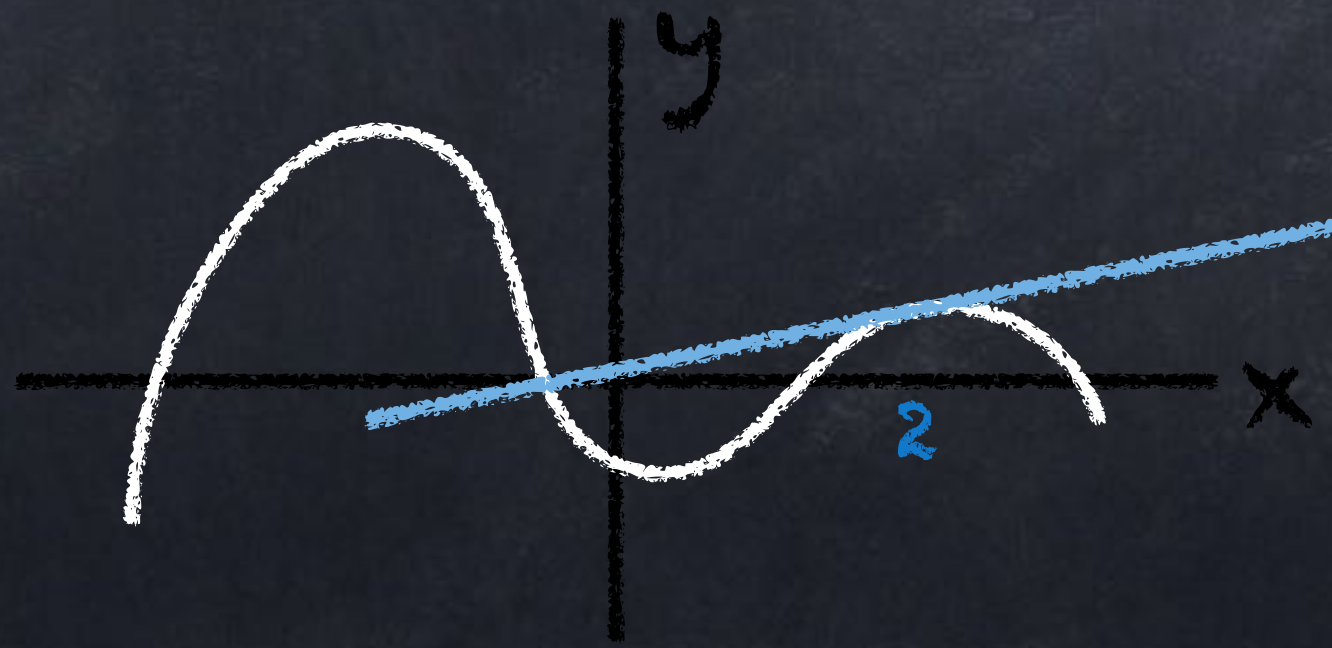


Velocity, slope, rate of change

- If we think of t as time and $f(t)$ as position, then $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$ is velocity when $t = 2$.



- If we think of x as horizontal and $f(x)$ as vertical, then $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$ is the slope of the tangent line to $y = f(x)$ at $x = 2$.



- No matter what f represents, $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$ is a rate of change of f.

Velocity, slope, rate of change

- No matter what f represents, $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ is a rate of change of f.
- Velocity is the rate of change of position with respect to time.
- Slope is the rate of change of y-position with respect to x-position.
- Acceleration is the rate of change of velocity with respect to time.
- Power is the rate of change of energy with respect to time.
- Current is the rate of change of charge with respect to time.
- Force is the rate of change of work with respect to position.
- Force is the rate of change of momentum with respect to time.
- Electric field is the rate of change of –voltage with respect to position.

Derivative at a point

The **derivative of $f(x)$ at $x = a$** (or the **derivative of f at a**) is the slope of line through the point $(a, f(a))$ that is tangent to the graph of f.

- $f'(a)$ spoken as "F prime of A" or "F prime at A"
- $\left.\dfrac{df}{dx}\right|_{x=a}$ spoken as "D F D X at X equals A" or "D F D X when X=A"
- $\left.\dfrac{dy}{dx}\right|_{x=a}$ spoken as "D Y D X at X equals A" or "D Y D X when X=A"

It is calculated as $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

Derivative at a point

The **derivative of $f(x)$ at $x = a$** (or the **derivative of f at a**) is calculated as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$